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NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS IN A BANACH SPACE

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ABSTRACT

We study the Cauchy problem associated with the Volterra integrodifferential equation

$$u'(t) \in Au(t) + \int_0^t B(t-s)u(s)ds + f(t), \quad u(0) = u_0 \in D(A),$$

where A is an m-dissipative non-linear operator (or more generally, an m-$\mathscr{D}(\omega)$ operator), defined on $D(A) \subset X$, where X is a real reflexive Banach space. We show that if B is of the form $B = FA + K$, where $F, K : X \to D(D_s)$, where D_s is the differentiation operator, with F bounded linear and K and D_sK Lipschitz continuous, then the Cauchy problem is well-posed. In addition we obtain an approximation result for the Cauchy problem.

1. Introduction

We consider the Cauchy problem associated with the Volterra integrodifferential equation

$$\text{(VE)} \qquad u'(t) \in Au(t) + \int_0^t B(t-s)u(s)ds + f(t), \quad u(0) = u_0 \in D(A),$$

where A is an m-dissipative nonlinear operator (or, more generally, an m-$\mathscr{D}(\omega)$ operator), defined on $D(A) \subset X$, where X is a real reflexive Banach space. This problem has been previously studied by a number of authors including Chen and Grimmer [5, 6], Crandall, Londen and Nohel [8], Crandall and Nohel [9], Miller [15], and Miller and Wheeler [16].

The approach we are using is to associate with (VE) an abstract nonlinear differential equation in a somewhat larger Banach space. It is then shown that the Cauchy problem for the Volterra integrodifferential equation (VE) is

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"well-posed" if and only if the Cauchy problem for the differential equation is "well-posed". Once this has been shown, one has available the nonlinear semigroup theory developed for nonlinear differential equations in a Banach space. In particular, the theory developed by Brezis and Pazy [4], Crandall [7], and Pazy [18] is useful in this regard as the operator C in the differential equation we examine is not dissipative, but rather $C-\omega I$ will be dissipative. While this is a trivial problem in the linear case (cf. Pazy [19]), it requires a great deal of effort in the nonlinear case to verify that the hoped for results are valid (cf. Pazy [18]). Besides the results obtained concerning the existence of a nonlinear semigroup, the "Trotter type" theorem developed by Brezis and Pazy [4] is particularly useful for our purposes. Using this result, we are able to obtain an approximation result for Volterra integrodifferential equations.

The association of (VE) with a differential equation much the same as the one used here was developed by Miller [15] for the linear case. The use of semigroup theory was then developed in Chen and Grimmer [5, 6] in the case when (VE) is a linear equation. Similar work in the linear case relating (VE) with a differential equation has also been carried out by Miller and Wheeler [16]. For the case of a linear integral equation, rather than an integrodifferential equation, related work has been done in Grimmer and Miller [12, 13].

Nonlinear semigroup theory has also been used in the study of nonlinear integral equations by Barbu [3] and Dafermos [10]. The approach we use here is different from that in [3] and [10] which is more related to the study of functional differential equations.

2. Preliminaries

We shall everywhere assume that the nonlinear operator A is in $m\text{-}\mathcal{D}(\omega)$; that is, that $A-\omega I$ is m-dissipative for some $\omega>0$, or alternately, $\omega I-A$ is m-accretive for some $\omega>0$, with domain $D(A)\subseteq X$ where X is a reflexive Banach space with norm $\|\ \ \|$. The main distinction between an m-dissipative operator and an operator in $m\text{-}\mathcal{D}(\omega)$ is that an m-dissipative operator generates a contraction semigroup while an operator in $m\text{-}\mathcal{D}(\omega)$ generates a quasi-contraction semigroup $\{T(t)\}$ which satisfies $\|T(t)x-T(t)y\|\leqq e^{\omega t}\|x-y\|$. In this case the semigroup $\{T(t)\}$ is said to be in Q_ω. For a further discussion of this matter see Barbu [2], Brezis and Pazy [4] and Crandall [7].

The function f is assumed to be defined on $[0,\infty)$ with values in X and is in the Sobolev space $W^{m,p}((0,\infty),X)$ of functions which together with their first m distributional derivatives are Bochner p-integrable functions, $p>1$. We further assume that $B(t)$ is defined on $D(A)$ and can be written as $B(t)x = F(t)Ax + K(t)x$. Here $F, K : X \to W^{m,p}((0,\infty), X)$ are defined by $(Fx)(t) = F(t)x$ and $(Kx)(t) = K(t)x$ and have the property that F is bounded linear and K is Lipschitz continuous. Further, we ask that F and K have range in the domain of D_s, $D(D_s)$, in $W^{m,p}((0,\infty), X)$ where D_s is the generator of the translation semigroup $\{S(t)\}$ on $W^{m,p}((0,\infty), X)$ defined by $(S(t)f)(s) = f(t+s)$. We further require that D_sK be Lipschitz continuous.

These conditions on B, while they appear to be quite stringent, will, in fact, allow a wide variety of possibilities. In particular, we shall be able to apply our results to the integrodifferential equation

$$u_t(t,x) = -A_1u(t,x) + \beta(u(t,x)) + \int_0^t [A_2(t-s,x)u(s,x) + \gamma(u(s,x))]ds + f(t,x)$$

where $A_1 = \Sigma_{|\alpha|\leq 2}\, a_\alpha(x)\partial_x^\alpha$ is a strongly elliptic second order partial differential operator with smooth coefficients, $A_2 = \Sigma_{|\alpha|\leq 2}\, b_\alpha(t,x)\partial_x^\alpha$ is any second order partial differential operator with smooth coefficients while β and γ are Lipschitz continuous functions.

Associated with (VE) will be the equation

$$\text{(DE)} \qquad z'(t) \in \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & B & D_s \end{bmatrix} z(t) \equiv Cz, \quad 0 < t < \infty,$$

where z is the transpose $(w, u, v)^*$ of (w, u, v) in $X \times X \times W^{m,p}((0,\infty), X) \equiv Z$. Also, $z(0) = z_0 \in D(C)$ where $D(C)$ is the domain of C in Z. On Z we use the norm $\|(w, u, v)\| = \|w\| + \|u\| + \|v\|_{m,p}$ where $\|\ \|_{m,p}$ is the norm on $W^{m,p}((0,\infty), X)$.

Remark 2.1. We note that since X is reflexive and $p > 1$, Z is also reflexive (cf., e.g., Adams [1]).

Definition 2.2. By a solution $u(t)$ of (VE) on $[0, T)$, $0 < T \leqq \infty$, we shall mean a continuous function $u : [0, T) \to X$ with $u(0) = u_0$, $u(t) \in D(A)$ a.e. such that $u(t)$ is locally Lipschitz and $u'(t) \in L^\infty((0, T), X)$. Further, there are functions v_1 and v_2 so that $v_i(t) \in Au(t)$ $(i = 1, 2)$ a.e. with $v_i \in L^\infty((0, T), X)$ $(i = 1, 2)$ and

$$u'(t) = v_1(t) + \int_0^t [F(t-s)v_2(s) + K(t-s)u(s)]ds + f(t), \quad \text{a.e.}$$

Definition 2.3. A function z defined on $[0, T)$ with values in Z is said to be a solution of (DE) if $z(t)$ is continuous in t on $[0, T)$ and Lipschitz on every compact interval of $[0, T)$, $z(0) = z_0$, $z(t) \in D(C)$ a.e. in $[0, T)$ and $z'(t) \in Cz(t)$ a.e. in $[0, T)$.

Our next proposition is central to our results and enables us to use semigroup theory.

PROPOSITION 2.4. *Suppose for* $z_0 = (w_0, u_0, v_0)^* \in D(C)$, (DE) *has a unique solution* $z(t) = (w(t), u(t), v(t))^*$ *on* $[0, T)$, $T < \infty$. *Then if* $f = v_0$, $u(t)$ *is a solution of* (VE) *on* $[0, T)$. *Conversely, if* $(w_0, u_0, v_0)^* \in D(C)$ *and* $u(t)$ *is a solution of* (VE) *with* $v_0 = f$ *then* $(w(t), u(t), v(t))^*$ *is a solution of* (DE) *with* $w(t) = w_0 + \int_0^t v_2(s)ds$ *and*

$$v(t)(s) = f(t+s) + \int_0^t [F(t-\tau+s)v_2(\tau) + K(t-\tau+s)u(\tau)]d\tau.$$

PROOF. If $z(t)$ is the unique solution of (DE) with $z_0 \in D(C)$ then we see that $w_0 \in X$, $u_0 \in D(A)$ and $v_0 \in D(D_s)$. Also, $w' \in Au$ is in $L^\infty((0, T), X)$ for each $T > 0$. Thus, $Fw' \in FAu$ and is in $L^\infty((0, T), W^{m,p}((0, \infty), X))$ as F is bounded linear. Similarly, D_sF is also bounded linear so that D_sFw' is also in $L^\infty((0, T), W^{m,p}((0, \infty), X))$. In fact, for $T < \infty$, Fw' and D_sFw' are in $L^1((0, T), W^{m,p}((0, \infty), X))$, as are Ku and D_sKu since K and D_sK are Lipschitzian. It now follows from Barbu [2; p. 32 Remark (j)] that the generalized solution

$$y(t) = S(t)v_0 + \int_0^t S(t-\tau)(Fw'(\tau) + Ku(\tau))d\tau$$

of the equation

$$y' = D_s y + Fw'(t) + Ku(t)$$

is in $W^{1,1}((0, T), W^{m,p}((0, \infty), X))$ and satisfies this equation almost everywhere in $(0, T)$. Also, since D_s generates the contraction semigroup $\{S(t)\}$, there can be at most one such solution.

We see that $(w(t), u(t), y(t))^*$ satisfies (DE) and so $y(t)$ must equal $v(t)$ for $t \geqq 0$ by uniqueness. Hence in $W^{m,p}((0, \infty), X)$,

$$v(t) = S(t)v_0 + \int_0^t S(t-\tau)(Fw'(\tau) + Ku(\tau))d\tau$$

and since $v(t) \in D(D_s)$, $v(t)(s)$ is absolutely continuous. So if $f(t) = v_0$ and $s \geqq 0$,

$$v(t)(s) = f(t+s) + \int_0^t [F(t-\tau+s)w'(\tau) + K(t-\tau+s)u(\tau)]d\tau.$$

In particular,

$$v(t)(0)=f(t)+\int_0^t [F(t-\tau)w'(\tau)+K(t-\tau)u(\tau)]d\tau.$$

Now since $u'\in Au+\delta_0 v$ a.e. in $(0,T)$,

$$u'(t)\in Au(t)+\int_0^t [F(t-\tau)w'(\tau)+K(t-\tau)u(\tau)]d\tau+f(t).$$

Thus,

$$u'(t)\in Au(t)+\int_0^t B(t-\tau)u(\tau)d\tau+f(t), \quad \text{a.e.},$$

$u(0)=u_0$, $u(t)\in D(A)$ a.e., u is continuous, Lipschitzian on every compact interval and $u'(t)\in L^\infty((0,T),X)$ for every $T<\infty$.

Now let $(w_0,u_0,v_0)^*\in D(C)$ and suppose u is a solution of (VE) with $v_0=f$. Since $v_2\in L^\infty$, $Fv_2(t)+Ku(t)$ is in $D(D_s)$ a.e. in t and $Fv_2(t)+Ku(t)$, $D_s(Fv_2(t)+Ku(t))$ are in $L^1((0,T),W^{m,p}((0,\infty),X))$. Thus the generalized solution

$$\text{(G.S.)} \qquad v(t)=S(t)v_0+\int_0^t S(t-\tau)(Fv_2(\tau)+Ku(\tau))d\tau$$

of $v'=D_s v+Fv_2(t)+Ku(t)$ is in $W^{1,1}((0,T),W^{m,p}((0,\infty),X))$ and satisfies the equation a.e. From (G.S.) we see that $v'\in L^\infty((0,T),W^{m,p}((0,\infty),X))$ and as $v_0=f$,

$$v(t)(s)=f(t+s)+\int_0^t (F(t-\tau+s)v_2(\tau)+K(t-\tau+s)u(\tau))d\tau$$

and

$$v(t)(0)=f(t)+\int_0^t (F(t-\tau)v_2(\tau)+K(t-\tau)u(\tau))d\tau.$$

Thus, u satisfies $u'=v_1(t)+\delta_0 v(t)$ a.e. Now define $w(t)=w_0+\int_0^t v_2(s)ds$. Then $w'\in L^\infty((0,T),X)$ and $(w,u,v)^*$ is a solution of (DE).

REMARK. If (DE) has the property that solutions are unique if they exist, then as $w_1(t)=w_0+\int_0^t v_1(s)ds$ is another choice we must have $v_1(t)=v_2(t)$ a.e.

Proposition 2.4 now allows us to examine the differential equation (DE) to obtain corresponding results for the equation (VE). In particular, if we show that C generates a semigroup $\{T(t)\}$ we know that for $z_0\in D(C)$, $T(t)z_0$ is the unique solution of (DE) with initial condition z_0. Further, if z_1 is also in $D(C)$, we will see that

$$\| T(t)z_0 - T(t)z_1 \| \leqq Me^{\omega t} \| z_0 - z_1 \|$$

where M and ω are positive constants. Thus, we obtain existence, uniqueness and continuity with respect to initial conditions for solutions of (VE) with $(0, u_0, f)^* \in D(C)$.

3. Main results

Our main results concern the well-posedness of (VE). In a later section we shall discuss a "Trotter type" theorem.

THEOREM 3.1. *Suppose A is in m-$\mathscr{D}(\omega)$, $B = FA + K$ where $F, K : X \to D(D_s)$ with F bounded linear and K and D_sK Lipschitz continuous. Then C generates a semigroup of nonlinear operators $\{T(t)\}$ and $T(t)z_0$ for $z_0 \in D(C)$ is the unique solution of* (DE).

THEOREM 3.2. *Suppose A is in m-$\mathscr{D}(\omega)$, $B = FA + K$ where $F, K : X \to D(D_s)$ with F bounded linear and K and D_sK Lipschitz continuous. Then* (VE) *is well-posed. That is,* (VE) *has a unique solution for u_0 and f where $(0, u_0, f)^* \in D(C)$ and if $(0, u_1, g) \in D(C)$ also, then if $u_1(t)$ is the unique solution of* (VE) *with u_0 and f replaced with u_1 and g,*

$$\| u(t) - u_1(t) \| \leqq Me^{\omega t} (\| u_0 - u_1 \| + \| f - g \|_{m,p})$$

where $\| \quad \|_{m,p}$ is the norm on $W^{m,p}((0, \infty), X)$.

EXAMPLE 1. An integrodifferential equation which satisfies the conditions of Theorem 3.2 is

$$u_t(t,x) = -Au(t,x) + \int_0^t a(t-s)Au(s,x)ds + f(t,x),$$

where $X = L^2(\Omega)$, $p > 1$, Ω a bounded domain in $\mathbf{R}^n$ with smooth boundary, $a(t) \in C_0^\infty(\mathbf{R}^+)$, and A is a nonlinear differential operator of the form

$$Au = \sum_{|\alpha| \leqq m} (-1)^\alpha D^\alpha A_\alpha (x, u, \cdots, D^m u),$$

where $A_\alpha(x, \xi)$ are real functions belonging to $C^\infty(\Omega \times \mathbf{R}^m)$ and satisfying the growth conditions

(1) $$|A_\alpha(x,\xi)| + C(|\xi| + g(x)) \qquad \text{for some } g \in L^2(\Omega)$$

and the monotonicity condition

$$\sum_{|\alpha|\leqq m} (A_\alpha(x,\xi)-A_\alpha(x,\eta))(\xi_\alpha-\eta_\alpha)\geqq 0. \tag{2}$$

Defining $a: H^m(\Omega)\times H^m(\Omega)\to R$ by

$$a(u,v)=\sum_{|\alpha|\leqq m}\int_\Omega A_\alpha(x,u,\cdots,D^m u)D^\alpha v dx$$

one obtains on the space V, $H_0^m(\Omega)\subset V\subset H^m(\Omega)$, a map $\tilde{A}: V\to V'$, the dual of V, which is maximal monotone. If one restricts $\tilde{A}$ so that

$$D(A)=\{u\in V:\tilde{A}u\in L^2(\Omega)\},\qquad Au=\tilde{A}u,\quad u\in D(A),$$

then A is maximal monotone on $L^2(\Omega)$ and, hence, m-accretive there. See Barbu [2, p. 49]. The rest of the conditions of Theorem 3.2 are easily verifiable.

Operators similar to A which lead to similar examples can be found in Lions [14].

EXAMPLE 2. Another integrodifferential equation which satisfies the conditions of Theorem 3.2 is

$$u_t(t,x)=-A_1u(t,x)+\beta(u(t,x))+\int_0^t[A_2(t-s,x)u(s,x)+\gamma(u(s,x))]ds+f(t,x)$$

where $X=L_2(\Omega)$, Ω a bounded domain in $\mathbf{R}^n$ with smooth boundary,

$$A_1=\sum_{|\alpha|\leqq 2}a_\alpha(x)\partial_x^\alpha,\qquad D(A_1)=H^2(\Omega)\cap H_0^1(\Omega),$$

is a strongly elliptic second order partial differential operator with coefficients $a_\alpha(x)\in C_0^\infty(\bar{\Omega})$, $A_2=\sum_{|\alpha|\leqq 2}b_\alpha(t,x)\partial_x^\alpha$ is any second order partial differential operator with coefficients $b_\alpha(t,x)\in C_0^\infty(\mathbf{R}^+\times\bar{\Omega})$ (more generally A_2 can be any second order pseudo-differential operator in the x-variables varying smoothly in t), and β and γ are Lipschitz continuous functions.

To show that the example is covered by Theorem 3.2, we first show that $-A_1+\beta$ is an m-$\mathscr{D}(\omega)$ operator. Indeed since $-(A_1+\lambda I)$, for some λ, generates a C_0 semi-group of contractions, it is m-dissipative. Hence $-A_1-\lambda I+\beta-\omega_1 I$ is also m-dissipative, where ω_1 is the Lipschitz constant of β. Thus $-A_1+\beta$ is an m-$\mathscr{D}(\omega)$ operator, where $\omega=\lambda+\omega_1$.

Next, we will demonstrate that $A_2+\gamma$ can be written in the form $F(A_1+\beta)+k$, where F is a bounded linear operator on $L_2(\Omega)$ and K is Lipschitz continuous. The requirement that F and K have range in $D(D_s)$ and that D_sK

be Lipschitz continuous will follow because of the smoothness of the coefficients of A_2.

Let $F = A_2L$, where L is a left parametrix of $-A_1$, i.e., $L(-A_1) = I + T$, where I is the identity and T is a pseudo-differential operator of order -2. L is also a pseudo-differential operator of order -2. (For the construction of L and T and the properties they satisfy, see L. Nirenberg [17].)

Then

$$\begin{aligned} F(-A_1+\beta) &= A_2L(-A_1+\beta) \\ &= A_2[L(-A_1)] + A_2L\beta \\ &= A_2(I+T) + A_2L\beta \\ &= A_2 + A_2T + A_2L\beta. \end{aligned}$$

Hence,

$$F(-A_1+\beta) + K = A_2 + \gamma,$$

where $K = -A_2T - A_2L\beta + \gamma$. What is left to prove is that F is bounded and K is Lipschitz continuous. Since γ and β are defined to be Lipschitz continuous, it suffices to show that A_2L and A_2T are bounded on $L_2(\Omega)$. This follows immediately from the fact that A_2L and A_2T are both pseudo-differential operators of order 0.

4. Proof of Theorem 3.1

We have identified with the initial value problem associated with the integrodifferential equation the abstract initial value problem

$$\text{(DE)} \qquad z' \in Cz, \qquad z(0) = z_0,$$

where

$$z = \begin{pmatrix} w \\ u \\ v \end{pmatrix} \in X \times X \times W^{m,p}((0,\infty), X) \quad \text{and} \quad C = \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & B & D_s \end{bmatrix}.$$

We can decompose the matrix C as follows:

$$C = \begin{bmatrix} 0, & A, & 0 \\ 0, & A, & 0 \\ 0, & FA - D_sF, & D_s \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_sF + K & 0 \end{bmatrix}$$

$$= P^{-1}C_1P \qquad\qquad + C_2$$

where

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & D_s \end{bmatrix}, \quad P = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F & I \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_sF + K & 0 \end{bmatrix}.$$

Hence

$$C = P^{-1}(C_1 + PC_2P^{-1})P.$$

We will prove that (DE) has a unique solution by showing that C generates a semi-group. We will demonstrate this after first showing that $C_1 + PC_2P^{-1} = PCP^{-1}$ belongs to $m\text{-}\mathcal{D}(\omega')$, where ω' is some constant to be determined later. We will prove this in a series of steps.

LEMMA 4.1. *$C_1 - \omega I$ is m-dissipative if A belongs to $m\text{-}\mathcal{D}(\omega)$.*

PROOF. Since D_s generates a contraction semi-group, D_s is m-dissipative. Hence, since A belongs to $m\text{-}\mathcal{D}(\omega)$, $C_1 - \omega I$ is m-dissipative.

We will next show that $PC_2P^{-1} - \omega_1 I$ is dissipative, where ω_1 is the Lipschitz constant of PC_2P^{-1}. In order to do this, we need the following lemma:

LEMMA 4.2. *$\delta_0(v)$ is bounded if $v \in W^{m,p}((0,\infty), X)$, $m \geqq 1$.*

PROOF. Let D_- be the left derivative. Then

$$D_-\|v\|^p = D_-(\|v\|^2)^{p/2}$$
$$= \frac{p}{2}(\|v\|^2)^{p/2-1}D_-(\|v\|^2).$$

Now $\|v\|^2 = \langle v, j(v)\rangle$, where $\langle \ , \ \rangle$ is the pairing of X and X^*, $\|\cdot\|$ is the norm of X, and $j(v)$ is a duality mapping. (See Deimling [11] for more details.) Then,

$$D_-\|v\|^p = \frac{p}{2}\|v\|^{p-2}D_-\langle v, j(v)\rangle$$
$$\leqq p\|v\|^{p-2}\langle v', j(v)\rangle.$$

Integrating, we have

$$\int_0^\infty \|v\|^{p-2}\langle v', j(v)\rangle dy \geqq \int_0^\infty D_-\frac{\|v\|^p}{p}\,dy = -\frac{\|v(0)\|^p}{p}.$$

Thus,

$$\frac{\|v(0)\|^p}{p} \leqq \int_0^\infty |\langle v', j(v)\|v\|^{p-2}\rangle|\,dy.$$

By Hölder's inequality, we then have

$$\|v(0)\|^p \leqq p\left(\int_0^\infty \|v'\|^p dy\right)^{1/p}\left(\int_0^\infty |j(v)\|v\|^{p-2}|^q dv\right)^{1/q},$$

if $1/p + 1/q = 1$. Hence

$$\|v(0)\|^p \leqq p\,\|v'\|_p\,\|v\|_p^{p/q},$$

since

$$\int_0^\infty |j(v)\|v\|^{p-2}|^q dv = \int_0^\infty \|v\|^{(p-1)q} dy = \int_0^\infty \|v\|^p dv.$$

This implies that

$$\begin{aligned}\|v(0)\| &\leqq p^{1/p}\,\|v'\|_p^{1/p}\|v\|_p^{1/q}\\ &\leqq p^{1/p}\left(\frac{\|v'\|_p}{p}+\frac{\|v\|_p}{q}\right).\end{aligned}$$

If we define $\|\cdot\|_{m,p}$ to be the norm associated with $W^{m,p}$, the last inequality implies that

$$\|\delta_0(v)\| = \|v(0)\| \leqq C\,\|v\|_{1,p}.$$

More generally, we have

$$\|\delta_0(v)\| \leqq C\,\|v\|_{m,p} \qquad \text{if } m \geqq 1.$$

LEMMA 4.3. *$PC_2P^{-1} - \omega_1 I$ is dissipative.*

PROOF. PC_2P^{-1} is shown to be bounded and Lipschitz continuous by appealing to the conditions imposed on K and F and the consequence of Lemma 4.2. Hence, if we choose ω_1 to be the Lipschitz constant of PC_2P^{-1}, we make $PC_2P^{-1} - \omega_1 I$ dissipative.

We are now able to show that $PCP^{-1} \in m\text{-}\mathcal{D}(\omega_1 + \omega)$.

LEMMA 4.4. *$C_1 + PC_2P^{-1} - (\omega + \omega_1)I$ is m-dissipative.*

PROOF. Let $S = C_1 - \omega I$ and $T = PC_2P^{-1} - \omega_1 I$. Since S and T are both dissipative, it suffices to show that $R(I - \lambda(S + T)) = Z$ for some $\lambda > 0$.

We consider the equation $[I - \lambda(S + T)]y \ni z$. We shall show that there exists a solution y for each $z \in Z$. Since S is m-dissipative, $R(I - \lambda S) = Z$. Hence $J_\lambda z = (I - \lambda S)^{-1}z$ exists for all $z \in Z$ and is non-expansive. Thus, if $[I - \lambda(S + T)]y \ni z$, then $(I - \lambda S)y - \lambda Ty \ni z$. This would imply that $y =$

$J_\lambda(\lambda Ty + z)$. To show that this y exists it suffices to show that the mapping $W_y = J_\lambda(\lambda Ty + z)$ has a fixed point, for every fixed z.

However,

$$Wy_1 - Wy_2 = J_\lambda(\lambda Ty_1 + z) - J_\lambda(\lambda Ty_2 + z).$$

This implies that

$$|Wy_1 - Wy_2| \leqq \lambda\, |Ty_1 - Ty_2|.$$

Since T is bounded and Lipschitz with Lipschitz constant M,

$$|Wy_1 - Wy_2| \leqq \lambda M\, |y_1 - y_2|.$$

Now choose λ so that $\lambda M < 1$. We have thus shown that W is a contraction mapping which ensures it of a fixed point.

PROPOSITION 4.5. *The initial-value problem*

$$r'(t) \in (C_1 + PC_2P^{-1})r(t), \qquad r(0) = r_0$$

has a unique solution $r(t) = S(t)r_0$, *where* $S(t)$ *is a semigroup in* $Q_{\omega+\omega_1}$.

PROOF. Since $C_1 + PC_2P^{-1} \in m\text{-}\mathcal{D}(\omega + \omega_1)$, by theorem 1.4 in Crandall [7] (see also the remark following chapter 3, theorem 1.3 in Barbu [2]),

$$S(t)r = \lim_{n\to\infty}\left(I + \frac{t}{n}(C_1 + PC_2P^{-1})\right)^{-n} r_0$$

exists for $t > 0$ for $u_0 \in D\overline{(C_1 + PC_2P^{-1})}$ and $S(t)$ belongs to $Q_{\omega+\omega_1}(\overline{D(C_1 + PC_2P^{-1})})$. $S(t)$ is defined on $\overline{X \times D(A) \times D(D_s)}$. Since X is reflexive, then so is $X \times X \times W^{m,p}((0,\infty), X)$. Hence, by corollary 1.1 of chapter 3 of Barbu [2], $r(t) = S(t)r_0$ is a unique solution of

$$r'(t) \in (C_1 + PC_2P^{-1})r(t), \qquad r(0) = r_0.$$

We are finally able to show that C generates a semi-group.

PROOF OF THEOREM 3.1. Consider the semi-group $T(t) = P^{-1}S(t)P$, where $S(t)$ is defined in Proposition 4.5. $T(t)$ is defined on $P^{-1}(\overline{X \times D(A) \times D(D_s)})$. However, P^{-1} maps $X \times D(A) \times D(D_s)$ into $X \times D(A) \times D(D_s)$. Hence $T(t)$ is actually defined on $\overline{X \times D(A) \times D(D_s)}$. To see this we note that

$$P^{-1} = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & F & I \end{bmatrix}.$$

Then

$$\begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & F & I \end{bmatrix}\begin{bmatrix} w \\ u \\ v \end{bmatrix} = \begin{bmatrix} w+u \\ u \\ Fu+v \end{bmatrix} = \begin{bmatrix} w_1 \\ u_1 \\ v_1 \end{bmatrix}.$$

If $w \in X$, $x \in D(A)$, $y \in D(D_s)$, then since $D(A) \subset X$, $w_1 = w + u \in X$, $u_1 = u \in D(A)$ and since $F : X \to D(D_S)$, $v_1 = Fu + v \in D(D_s)$.

Thus $T(t)$ is defined on $X \times \overline{D(A)} \times W^{m,p}((0,\infty), X)$ (since $\overline{D(D_s)} = W^{m,p}((0,\infty), X)$).

Now $r(t) = S(t)r_0$ satisfies

$$r' \in (C_1 + PC_2P^{-1})r, \qquad r(0) = r_0 = Pz_0. \tag{4.1}$$

If we let $z(t) = T(t)z_0$, then since $r(t) = Pz(t)$ is a unique solution to the Cauchy problem (4.1),

$$(Pz(t))' \in (C_1 + PC_2P^{-1})Pz(t), \qquad Pz(0) = r_0. \tag{4.2}$$

Since P is linear and constant, (4.2) implies that

$$z'(t) \in P^{-1}(C_1 + PC_2P^{-1})Pz(t), \qquad z(0) = P^{-1}r_0 = z_0.$$

Hence $z(t) = T(t)z_0$ is a unique solution to the Cauchy problem

$$z' \in Cz, \qquad z(0) = z_0.$$

Finally, $T(t)$ is a semi-group satisfying

$$\begin{aligned} \|T(t)y_0 - T(t)y_1\| &= \|P^{-1}S(t)Py_0 - P^{-1}S(t)Py_1\| \\ &\leqq \|P^{-1}\| \, \|S(t)Py_0 - S(t)Py_1\| \\ &\leqq \|P^{-1}\| \, e^{(\omega_1+\omega)t} \|Py_0 - Py_1\| \\ &\leqq \|P^{-1}\| \, \|P\| \, e^{(\omega_1+\omega)t} \|y_0 - y_1\| \\ &\leqq Me^{(\omega_1+\omega)t} \|y_0 - y_1\|. \end{aligned}$$

5. Approximations

In this section we consider the equations

$$\begin{aligned} &u_n'(t) \in A_n u_n(t) + \int_0^t B_n(t-s)u_n(s)ds + f(t), \\ &u_n(0) = u_0 \in D(A_n) = D(A). \end{aligned} \tag{VE$_n$}$$

Here $B_n = F_nA_n + K_n$ where F_n and K_n have range in the domain of D_s, K_n and D_sK are Lipschitzian while F_n is bounded linear on X. Also, $A_n \in m\text{-}\mathcal{D}(\omega_n)$ and $f \in W^{m,p}((0,\infty), X)$. It follows from our previous work that (VE_n) is well posed. We are able to prove the following theorem related to theorem 6.3 of [6].

THEOREM 5.1. *Assume that* $\|F_n - F\| \to 0$, $\|D_sF_n - D_sF\| \to 0$ *and* $K_n \to K$ *uniformly on bounded sets. Further, assume* $A_n \in m\text{-}\mathcal{D}(\omega_n)$ *and* $A \in m\text{-}\mathcal{D}(\omega)$ *where* $0 \leqq \omega_n, \omega < \alpha < \infty$ *for some constant* α. *Suppose also that there exists* $\lambda_0 > 0$ *so that if* $J^n_\lambda = (I - \lambda A_n)^{-1}$ *and* $J_\lambda = (I - \lambda A)^{-1}$, $J^n_\lambda u \to J_\lambda u$ *for every* $u \in X$ *and* $0 < \lambda < \lambda_0$. *If* $u_n(t)$ *is the solution of* (VE_n) *then* $\lim_{n\to\infty} u_n(t) = u(t)$ *uniformly on compact* t *intervals.*

PROOF. Consider the differential equations

$$(\text{DE}_n) \qquad z'_n \in C^n z_n, \qquad z_n(0) = z_0$$

and the operators

$$P_n = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F_n & 0 \end{bmatrix}, \qquad P_n^{-1} = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & F_n & 0 \end{bmatrix},$$

$$C^n_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_sF_n + K_n & 0 \end{bmatrix}, \qquad C^n_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & D_s \end{bmatrix},$$

$$C^n = P_n^{-1}(C^n_1 + P_nC^n_2P_n^{-1})P_n.$$

Now, to show that z_n converges to z uniformly on compact t intervals we shall consider instead the problems

$$r'_n \in (C^n_1 + P_nC^n_2P_n^{-1})r_n, \qquad r' \in (C_1 + PC_2P^{-1})r.$$

We shall show that $r_n \to r$ on compact t intervals and then since $r_n = S_nz_0$ and $T_n = P_n^{-1}S_nP_n$, we will show $z_n \to z$ on compact t intervals. We relabel $P_nC^n_2P_n^{-1}$ to be C^n_3 and note that $(I - \lambda C^n_1)^{-1}$ is given by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & (I - \lambda A_n)^{-1} & 0 \\ 0 & 0 & (I - \lambda D_s)^{-1} \end{bmatrix}$$

so that by hypothesis, $(I - \lambda C^n_1)^{-1}z_0 \to (I - \lambda C_1)^{-1}z_0$ for every $z \in Z$. Now C^n_3 is bounded uniformly so that $C^n_1 + C^n_3$ is in $\mathcal{D}(\alpha_1)$ for some constant $\alpha_1 > 0$ and $(I - \lambda C^n_1 - \lambda C^n_3)^{-1}$ exists as a single valued operator and is given by

$$(I - \lambda C^n_1)^{-1}(I - \lambda C^n_3(I - \lambda C^n_1)^{-1})^{-1} \quad \text{or} \quad J^n_{\lambda,1}(I - \lambda C^n_3 J^n_{\lambda,1})^{-1}$$

where $J^n_{\lambda,1} = (I - \lambda C^n_1)^{-1}$. For later convenience we define $J^0_{\lambda,1} = (I - \lambda C_1)^{-1}$. Now as $J^n_{\lambda,1}$ is Lipschitzian with Lipschitz constant $(1 - \lambda\alpha)^{-1}$ because $C^n_1 \in \mathcal{D}(\alpha)$, we see that $(I - \lambda C^n_3 J^n_{\lambda,1})^{-1}$ exists for $0 < \lambda < \lambda_0$ where $\lambda_0 C_5(1 - \lambda_0\alpha)^{-1} < 1$ and C_5 is a Lipschitz constant for C^n_3 for all n. Indeed, $(I - \lambda C^n_3 J^n_{\lambda,1})^{-1} z = g_n$ or $z = (I - \lambda C^n_3 J^n_{\lambda,1}) g_n$ is uniquely solvable for g_n given z since $T^n_{\lambda,1} g = z + \lambda C^n_3 J^n_{\lambda,1} g$ is a contraction map with uniform contraction constant $\lambda_0 C_5(1 - \lambda_0\alpha)^{-1} \equiv \alpha_2$. Thus, for each n there is unique g_n such that

$$g_n = z + \lambda C^n_3 J^n_{\lambda,1} g_n \quad \text{or} \quad g_n = (I - \lambda C^n_3 J^n_{\lambda,1})^{-1} z.$$

We note that $g_n \to g_0$. Indeed,

$$\begin{aligned} \|g_n - g_0\| &= \|\lambda C^n_3 J^n_{\lambda,1} g_n - \lambda C^0_3 J^0_{\lambda,1} g_0\| \\ &\leqq \|\lambda C^n_3 J^n_{\lambda,1} g_n - \lambda C^n_3 J^n_{\lambda,1} g_0\| + \|\lambda C^n_3 J^n_{\lambda,1} g_0 - \lambda C^0_3 J^0_{\lambda,1} g_0\| \\ &\leqq \alpha_2 \|g_n - g_0\| + \|\lambda C^n_3 J^n_{\lambda,1} g_0 - \lambda C^n_3 J^0_{\lambda,1} g_0\| + \|\lambda C^n_3 J^n_{\lambda,1} g_0 - \lambda C_3 J^0_{\lambda,1} g_0\|. \end{aligned}$$

Hence,

$$\|g_n - g_0\| \leqq (1 - \alpha_2)^{-1} \lambda C_5 \|J^n_{\lambda,1} g_0 - J^0_{\lambda,1} g_0\| + (1 - \alpha_2)^{-1} \lambda \, \|C^n_3 J^0_{\lambda,1} g_0 - C_3 J^0_{\lambda,1} g_0\|$$

and since $J^n_{\lambda,1} g_0 \to J^0_{\lambda,1} g_0$ and $C^n_3 v \to C_3 v$ for all $v \in Z$ we see that $g_n \to g_0$ as $n \to \infty$. That is,

$$(I - \lambda C^n_3 J^n_{\lambda,1})^{-1} z \to (I - \lambda C_3 J^0_{\lambda,1})^{-1} z.$$

It now follows that

$$J^n_{\lambda,1}(I - \lambda C^n_3 J^n_{\lambda,1})^{-1} z \to J^0_{\lambda,1}(I - \lambda C_3 J^0_{\lambda,1})^{-1} z$$

or

$$(I - \lambda C^n_1 - \lambda C^n_3)^{-1} z \to (I - \lambda C_1 - \lambda C_3)^{-1} z.$$

Now as $C^n_1 + C^n_3$ and $C_1 + C_3$ are in $\mathcal{D}(\alpha_1)$ it follows that $S_n(t) z_0 \to S(t) z_0$ (cf. [4, theorem 3.1]). Hence, $T_n(t) z_0 = P_n^{-1} S_n(t) P_n z_0 \to P^{-1} S(t) P z_0 = T(t) z_0$. This concludes the proof since $T_n(t) z_0 = (w_n(t), u_n(t), v_n(t))$ and $T(t) z_0 = (w(t), u(t), v(t))$.

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